DATE OF EXAM: 26.02.2018 **Solution**

SUBJECT NAME: Analysis II - MIDTERM Exam - Semester II

1. *Compute upper and lower Riemann integrals of following function and determine as to whether it is Riemann integrable or not: Let* $f : [0, 2] \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} 5 & \textit{if } x^2 \textit{ is rational.} \\ 7 & \textit{otherwise.} \end{cases}$$

Solution: Note that square of irrational number on the real line need not be rational. Let $P = \{0 = t_0 < t_1 < ... < t_n = b\}$ be any partition of $[0, 2]$.

The upper sum is

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1} - t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} 7 \cdot (t_k - t_{k-1}) = 7 \cdot 2 = 14.$$

The lower sum is

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1} - t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} 5 \cdot (t_k - t_{k-1}) = 5 \cdot 2 = 10.$$

The upper Riemann integral is $U(f) = 14$ and the lower Riemann integral is $L(f) = 10$. Thus $U(f) \neq L(f)$, f is not Riemann integrable.

□

2. *Let* $a < b$ *be real numbers. Suppose* $f : [a, b] \to \mathbb{R}$ *is monotonic. Show that* f *is Riemann integrable.*

Solution: We can find the proof in the book 'Elementary Analysis' by Kenneth A. Ross. Theorem 33.1, Page - 280.

□

3. *Let* $a < b$ *and* $c < d$*, be real numbers and let* $u : [a, b] \to [c, d]$ *be a continuously differentiable function. Let* $f : [c, d] \to \mathbb{R}$ *be a continuous function. Show that*

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(y)dy.$$

Solution: We can find the proof in the book 'Elementary Analysis' by Kenneth A. Ross. Theorem 34.4, Page - 295.

□

4. *Define* $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ *by* $d(m,n) = |\frac{1}{m^2} - \frac{1}{n^2}|$. *Show that* d *defines a metric on* $\mathbb{N}$. *Show that* $\mathbb{N}$ *complete with respect to this metric.*

Solution: (i) If $d(m,n) = 0$ then $|\frac{1}{m^2} - \frac{1}{n^2}| = 0$, $m^2 = n^2$, implies $m = n$.
(ii) $d(m,n) = |\frac{1}{m^2} - \frac{1}{n^2}| = |\frac{1}{n^2} - \frac{1}{m^2}| = d(n,m)$
(iii) $d(m,n) = |\frac{1}{m^2} - \frac{1}{n^2}| = |\frac{1}{m^2} - \frac{1}{k^2} + \frac{1}{k^2} - \frac{1}{n^2}| \le |\frac{1}{m^2} - \frac{1}{k^2}| + |\frac{1}{k^2} - \frac{1}{n^2}| = d(m,k) + d(k,n)$. Therefore d is metric.

Consider a sequence $(x_n = n)$, we will prove that (x_n) is Cauchy sequence. Let $\epsilon > 0$. Since $\frac{1}{n^2}$ converges to 0 with respect to the usual topology in $\mathbb{R}$, we have the following assertion:

$$\exists \ n_0 \in \ \mathbb{N} \ \text{such that} \ \forall \ \ m,n \ge n_0, \quad \frac{1}{n^2} < \epsilon \ \text{ and } \ \frac{1}{m^2} < \epsilon.$$

Now $\forall \ \ m,n \ge n_0$, we have

$$d(n,m) = |\frac{1}{n^2} - \frac{1}{m^2}| \le |\frac{1}{n^2}| + |\frac{1}{m^2}| < \epsilon + \epsilon = 2\epsilon.$$

Thus (x_n) is Cauchy sequence.

Now, we will prove that (x_n) does not converges to a, for all $a \in \mathbb{N}$. Since $d(x_n, a) = |\frac{1}{n^2} - \frac{1}{a^2}| \to \frac{1}{a^2} \neq 0$, therefore x_n does not converges to a.

□

5. *On* $\mathbb{R}^2$, *consider the usual metric* d, *defined by*

$$d((x_1,x_2),(y_1,y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

For the following subsets A, B *of* $\mathbb{R}^2$, *determine the closure and the interior of the closure.*

(i) $A = \{(x_1,x_2) : x_1 + x_2$ *is rational* $\}$.

(ii) $B = \{(x_1,x_2) : |x_1| < |x_2|\}$.

Solution: (i) Since $\mathbb{Q} \times \mathbb{Q} \subset A$, we have $\mathbb{R}^2 = \overline{\mathbb{Q} \times \mathbb{Q}} \subset \overline{A}$. Therefore $\overline{A} = \mathbb{R}^2$. Since $\overline{A} = \mathbb{R}^2$ is open in $\mathbb{R}^2$, we have $(\overline{A})^o = (\mathbb{R}^2)^o = \mathbb{R}^2$.

(ii)Let $S = \{(x_1,x_2) : |x_1| \le |x_2|\}$, we claim that $\overline{B} = S$. Let $x = (x_1,x_2) \in S$, if $|x_1| < |x_2|$ then $x \in B$, therefore $x \in \overline{B}$. Suppose that $|x_1| = |x_2|$ and let $r > 0$ arbitrary. We will show that $B(x,r) \cap B \neq \emptyset$. For this, choose $0 < \delta < r$ such that the closed rectangle $[x_1 - \delta, x_1 + \delta] \times [x_2 - \delta, x_2 + \delta] \subset B(x,r)$. Now, take $y = (x_1 - \delta, x_2)$, since $|x_1 - \delta| < |x_1| = |x_2|$, we have $y \in B$. Clearly $y \in B(x,r)$ and therefore $B(x,r) \cap B \neq \emptyset$. This implies that $x \in \overline{B}$. Thus $S \subset \overline{B}$. We know that S is closed in $\mathbb{R}^2$ and contains B, therefore $\overline{B} \subset S$. Hence $\overline{B} = \{(x_1,x_2) : |x_1| \le |x_2|\} = S$.

Now we claim that $(\overline{B})^o = B$. Since B is open, we have $B = B^o \subset (\overline{B})^o$. If $x = (x_1,x_2)$ with $|x_1| = |x_2|$ then for any $r > 0$, we can choose $0 < \delta < r$ such that the closed rectangle $[x_1 - \delta, x_1 + \delta] \times [x_2 - \delta, x_2 + \delta] \subset B(x,r)$. Now, take $y = (x_1 + \delta, x_2)$. Clearly $y \in B(x,r)$, but $y \notin \overline{B}$. Thus, for every $r > 0$, we have $B(x,r) \not\subset \overline{B}$. This implies that $\overline{B}^o = B = \{(x_1,x_2) : |x_1| < |x_2|\}$.

□

6. *Denote interior of a subset S of a metric space by S^o. Let C, D be subsets of a metric space (Y, d). Show that $(C \bigcup D)^o \supseteqq C^o \bigcup D^o$. Give an example where $(C \bigcup D)^o \neq C^o \bigcup D^o$.*

   Solution: Let $x \in C^o \bigcup D^o$ then either $x \in C^o$ or $x \in D^o$. There exists $r > 0$ such that either $B(x, r) \subset C$ or $B(x, r) \subset D$. Therefore $B(x, r) \subset C \bigcup D$, hence $x \in (C \bigcup D)^o$.

   Let $\mathbb{R}$ with usual metric and let $C = \mathbb{Q}$ and D is set of irrationals. Then $C^o = \emptyset$ and $D^o = \emptyset$, therefore $C^o \bigcup D^o = \emptyset$. $C \bigcup D = \mathbb{R}$, thus $(C \bigcup D)^o = \mathbb{R}$.

   $\square$

7. *Let (X, d_1) be a metric space. Define d_2 on $X \times X$ by*

$$d_2(x, y) = \begin{cases} d_1(x, y) & \text{if } 0 \leq d_1(x, y) \leq 1 \\ 1 & \text{Otherwise.} \end{cases}$$

   Show that d_2 is a metric on X. Show that a set A is open in (X, d_1) iff it is open in (X, d_2).

   Solution:(i).Let $x, y \in X$, if $d_2(x, y) = 0$ then $d_1(x, y) = 0$, therefore $x = y$.
   (ii). Let $x, y \in X$, if $d_1(x, y) \leq 1$ then $d_2(x, y) = d_1(x, y) = d_1(y, x) = d_2(y, x)$. If $d_1(x, y) > 1$ then $d_1(y, x) > 1$, therefore $d_2(x, y) = d_2(y, x)$.
   (iii). Let $x, y, z \in X$, If $d_1(x, y) \leq 1$ and $d_1(y, z) \leq 1$ then $d_2(x, y) = d_1(x, y)$ and $d_2(y, z) = d_1(y, z)$. Therefore $d_2(x, z) \leq d_1(x, z) \leq d_1(x, y) + d_1(y, z) = d_2(x, y) + d_2(y, z)$.
   If $d_1(x, y) > 1$, then $d_2(x, z) \leq 1 \leq 1 + d_1(y, z) = d_2(x, y) + d_2(y, z)$. Similarly we can prove the other cases. Hence d_2 is a metric on X.

   Let U be d_1-open and $x \in U$, then there exits $r > 0$ such that $B_{d_1}(x, r) \subset U$. If this holds for r, then it holds for any $0 < \epsilon < r$. So we may assume that $0 < r < 1$. In such case $B_{d_1}(x, r) = B_{d_2}(x, r)$, so that $B_{d_2}(x, r) \subset U$. Therefore U is d_2- open. Similarly we can prove that if U is d_2-open then U is d_1-open.

   $\square$